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To Express the Roots of the Solvable Quantics as Symmetrical Functions of Homologues.

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1. The quadric
$$0 = ax^2 + 2bx + c$$
$$= (ax + b)^2 - (b^2 - ac) \tag{1}$$
$$= (ax + b)^2 - |b^2|, \tag{2}$$
where
$$b^2 - ac = |b^2|, \text{ the discriminant of the quadric.} \tag{3}$$
Assume
$$0 = ax + b + b', \tag{4}$$
then
$$0 = (ax + b)^2 - b'^2. \tag{5}$$

Comparing this with (2), we have
$$|b^2| = b', \tag{6}$$
whence
$$b' = \pm |b^2|^{\frac{1}{2}}. \tag{7}$$

The complete solution of the quadric is then given by the system
$$0 = ax_1 + b + b', \tag{8_1}$$
$$0 = ax_2 + b - b'. \tag{8_2}$$
If the discriminant
$$\Delta_2 = \begin{vmatrix} a, & b \\ b, & c \end{vmatrix} = ac - b^2 = -|b^2| \tag{9}$$
is positive, the roots are complex conjugates. Placing
$$\cos\beta = \frac{b}{\sqrt{ac}}, \text{ then } b = \sqrt{ac}\cos\beta;\ b' = i\sqrt{ac}\sin\beta, \tag{10}$$
and the quadric is solved by the system
$$0 = a^{\frac{1}{2}}x_1 + c^{\frac{1}{2}}\varepsilon^{\beta i} = a^{\frac{1}{2}}x_1\varepsilon^{-\beta i} + c^{\frac{1}{2}}, \tag{$8_1'$}$$
$$0 = a^{\frac{1}{2}}x_2 + c^{\frac{1}{2}}\varepsilon^{-\beta i} = a^{\frac{1}{2}}x_2\varepsilon^{\beta i} + c^{\frac{1}{2}}. \tag{$8_2'$}$$

2. The cubic

$$\begin{aligned} 0 &= ax^3 + 3bx^2 + 3cx + d \\ &= (ax+b)^3 - 3\,(b^2 - ac)(ax+b) + 2\,(b^3 - \tfrac{3}{2}abc + \tfrac{1}{2}a^2d) \qquad (11) \\ &= (ax+b)^3 - 3\,|b^2|\,(ax+b) + 2\,|b^3|, \qquad (12) \end{aligned}$$

where

$$b^3 - \tfrac{3}{2}abc + \tfrac{1}{2}a^2d = |b^3|, \text{ the cubic variant.} \tag{13}$$

Assume

$$0 = ax + b + b' + b'', \tag{14}$$

then we have

$$0 = (ax+b)^3 - 3b'b''\,(ax+b) + b'^3 + b''^3, \tag{15}$$

and comparing this with (12), we have

$$|b^2| = b'b'', \tag{16}$$

$$|b^3| = \tfrac{1}{2}(b'^3 + b''^3), \tag{17}$$

therefore

$$b' = \{\,|b^3| + (|b^3|^2 - |b^2|^3)^{\frac{1}{2}}\}^{\frac{1}{3}}, \tag{18'}$$

$$b'' = \{\,|b^3| - (|b^3|^2 - |b^2|^3)^{\frac{1}{2}}\}^{\frac{1}{3}}. \tag{18''}$$

Except in the irreducible case,* these formulæ give the numerical values of the auxiliaries b', b'', which are obviously homologues to b, and being used in (14), give one root and the complete solution is given by the system

$$0 = ax_1 + b + b' + b'', \tag{19$_1$}$$

$$0 = ax_2 + b + b'1^{\frac{1}{3}} + b''1^{-\frac{1}{3}}, \tag{19$_2$}$$

$$0 = ax_3 + b + b'1^{-\frac{1}{3}} + b''1^{\frac{1}{3}}, \tag{19$_3$}$$

if

$$1^{\frac{1}{3}} = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3};\ 1^{-\frac{1}{3}} = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}. \tag{20}$$

For the reciprocal solution we have

$$\begin{aligned} 0 &= ax^3 + 3bx^2 + 3cx + d, \\ &= 2\,(\tfrac{1}{2}ad^2 - \tfrac{3}{2}bcd + c^3) - 3\,(c^2 - bd)(c + dx^{-1}) + (c + dx^{-1})^3 \qquad (21) \\ &= 2\,|c^3| - 3\,|c^2|\,(c + dx^{-1}) + (c + dx^{-1})^3, \qquad (22) \end{aligned}$$

where

$$c^2 - bd = |c^2|, \quad \text{the quadric retrovariant,} \tag{23}$$

$$c^3 - \tfrac{3}{2}bcd + \tfrac{1}{2}a^2d = |c^3|, \quad \text{`` cubic ``} \tag{24}$$

Assuming the system

$$0 = c'' + c' + c + dx_1^{-1}, \tag{25$_1$}$$

$$0 = c''1^{\frac{1}{3}} + c'\,1^{-\frac{1}{3}} + c + dx_2^{-1}, \tag{25$_2$}$$

$$0 = c''1^{-\frac{1}{3}} + c'1^{\frac{1}{3}} + c + dx_3^{-1}, \tag{25$_3$}$$

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* See my paper, " Symmetries of the Cubic and Methods of Treating the Irreducible Case " (Annals of Mathematics).

where the cube root of unity factors in the corresponding forms of (19) and (25) have been chosen so that their product is 1. This arrangement, though arbitrary, is adopted because it produces the most perfect symmetry. We have then

$$0 = c''^3 + c'^3 - 3c'c''(c + dx^{-1}) + (c + dx^{-1})^3. \tag{26}$$

Comparing this with (22), we have

$$|c^2| = c'c'', \tag{27}$$

$$|c^3| = \tfrac{1}{2}(c'^3 + c''^3), \tag{28}$$

therefore

$$c' = \{ |c^3| \pm (|c^3|^2 - |c^2|^3)^{\frac{1}{2}} \}^{\frac{1}{3}}, \tag{29$'$}$$

$$c'' = \{ |c^3| \mp (|c^3|^2 - |c^2|^3)^{\frac{1}{2}} \}^{\frac{1}{3}}, \tag{29$''$}$$

where the sign must be left indeterminate, not having given yet the connecting condition between auxiliaries b', b'' and c', c''.

Regarding the condition $0 = 1 + 1^{\frac{1}{3}} + 1^{-\frac{1}{3}}$, we easily deduce from (19)

$$0 = \frac{a}{3}(x_1 + x_2 + x_3) + b. \tag{30}$$

$$0 = \frac{a}{3}(x_1 + x_1 1^{-\frac{1}{3}} + x_3 1^{\frac{1}{3}}) + b', \tag{30$'$}$$

$$0 = \frac{a}{3}(x_1 + x_2 1^{\frac{1}{3}} + x_3 1^{-\frac{1}{3}}) + b''. \tag{30$''$}$$

Similarly from (25),

$$0 = c'' + \frac{d}{3}(x_1^{-1} + x_2^{-1} 1^{-\frac{1}{3}} + x_3 1^{\frac{1}{3}}) = c'' - \frac{a}{3}(x_2 x_3 + x_3 x_1 1^{-\frac{1}{3}} + x_1 x_2 1^{\frac{1}{3}}), \tag{31$''$}$$

$$0 = c' + \frac{d}{3}(x_1^{-1} + x_2^{-1} 1^{\frac{1}{3}} + x_3 1^{-\frac{1}{3}}) = c' - \frac{a}{3}(x_2 x_3 + x_3 x_1 1^{\frac{1}{3}} + x_1 x_2 1^{-\frac{1}{3}}), \tag{31$'$}$$

$$0 = c + \frac{d}{3}(x_1^{-1} + x_2^{-1} + x_3^{-1}) = c - \frac{a}{3}(x_2 x_3 + x_3 x_1 + x_1 x_2). \tag{31}$$

From these we deduce the remarkable symmetries*

$$b'b'' = b^2 - ac = |b^2|, \text{ same as (16)}, \tag{32}$$

$$b''b = b'^2 - ac', \tag{32$'$}$$

$$bb' = b''^2 - ac'', \tag{32$''$}$$

$$bc + b'c' + b''c'' = ad, \tag{33}$$

$$bc' + b'c'' + b''c = 0, \tag{33$'$}$$

$$bc'' + b'c + b''c' = 0, \tag{33$''$}$$

$$c'c = c''^2 - b''d, \tag{34$''$}$$

$$cc'' = c'^2 - b'd, \tag{34$'$}$$

$$c''c' = c^2 - bd = |c^2|, \text{ same as (27)}. \tag{34}$$

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*Given first in my paper, "Symmetries of the Cubic," etc.

Relation (33) has its symmetrical form by virtue of the assumption of cube root of unity factors in (19) and (25), and gives therefore the condition connecting the auxiliaries b', b'' with c', c'' and placing

$$\tfrac{1}{2}(ad-bc)=\tfrac{1}{2}(b'c'+b''c'')=|bc|,\ \text{the connectant,}^{*} \qquad (35)$$
$$p'=b'c'\,;\ p''=b''c'',$$

we obtain p' and p'' as roots of the quadric resolvent

$$0=p^2-2|bc|p+|b^2|.|c^2|, \qquad (36)$$

whence

$$p'=b'c'\ =|bc|+\sqrt{|bc|^2-|b^2|.|c^2|}, \qquad (37')$$
$$p''=b''c''=|bc|-\sqrt{|bc|^2-|b^2|.|c^2|}. \qquad (37'')$$

From (32) we deduce

$$\begin{aligned} bb'b'' &= b^3-abc,\\ &= b'^3-ab'c'\ \therefore\ b'=\sqrt[3]{b|b^2|+ap'}, && (38')\\ &= b''^3-ab''c''\ \therefore\ b''=\sqrt[3]{b|b^2|+ap''}. && (38'')\end{aligned}$$

Similarly from (34),

$$\begin{aligned} c''c'c &= c''^3-b''c''d\ \therefore\ c''=\sqrt[3]{c|c^2|+dp''} && (39'')\\ &= c'^3-b'c'd\ \therefore\ c'=\sqrt[3]{c|c^2|+dp'} && (39')\\ &= c^3-bcd. \end{aligned}$$

The discriminant of the cubic is

$$\Delta_3=\begin{vmatrix} a, & 2b, & c, & 0\\ 0, & a, & 2b, & c\\ b, & 2c, & d, & 0\\ 0, & b, & 2c, & d \end{vmatrix}=4\begin{vmatrix} |bc|, & |b^2|\\ |c^2|, & |bc| \end{vmatrix}\begin{aligned} &=4(|bc|^2-|b^2|.|c^2|)=\frac{4}{a^2}(|b^3|^2-|b^2|^3)\\ &=\frac{4}{d^2}(|c^3|^2-|c^2|^3)\\ &=(b'c'-b''c'')^2=(p'-p'')^2, \quad (40)\end{aligned}$$

and representing the coefficients of the cubic in terms of the auxiliaries b', b'', c', c'', we have

$$a=\frac{b'^3-b''^3}{b'c'-b''c''}=\Delta_3^{-\frac{1}{2}}(b'^3-b''^3), \qquad (41)$$

$$b=-\frac{b'^2c''-b''^2c'}{b'c'-b''c''}=-\Delta_3^{-\frac{1}{2}}(b'^2c''-b''^2c'), \qquad (42)$$

$$c=\frac{b'c''^2-b''c'^2}{b'c'-b''c''}=\Delta_3^{-\frac{1}{2}}(b'c''^2-b''c'^2), \qquad (43)$$

$$d=-\frac{c''^3-c'^3}{b'c'-b''c''}=-\Delta_3^{-\frac{1}{2}}(c''^3-c'^3). \qquad (44)$$

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* No name ever having been given to this important form, this name is thought to be suggestive of its function.

We thus obtain the cubic in the canonical form

$$\begin{aligned} 0 &= \Delta_3^{-\frac{1}{2}}\{(b'^3 - b''^3)\,x^3 - 3\,(b'^2c'' - b''^2c')\,x^2 + 3\,(b'c''^2 - b''c'^2)\,x - (c''^3 - c'^3)\} \\ &= \Delta_3^{-\frac{1}{2}}\{(b'x - c'')^3 - (b''x - c')^3\}, \end{aligned} \tag{45}$$

whence the following system of root forms:

$$0 = b'x_1 - c'' \quad - (b''x_1 - c'), \tag{46$_1$}$$

$$0 = (b'x_2 - c'')\,1^{\frac{1}{3}} - (b''x_2 - c')\,1^{-\frac{1}{3}}, \tag{46$_2$}$$

$$0 = (b'x_3 - c'')\,1^{-\frac{1}{3}} - (b''x_3 - c')\,1^{\frac{1}{3}}. \tag{46$_3$}$$

If the discriminant is negative, then the auxiliaries b', b'', c', c'' are complex conjugates. In that case assume in (12)

$$|b^3| = |b^2|^{\frac{3}{2}} \cos\phi \;\therefore\; \cos\phi = |b^3|\,.\,|b^2|^{-\frac{3}{2}}, \tag{47}$$

$$0 = ax + b + 2\,|b^2|^{\frac{1}{2}} \cos\tfrac{1}{3}\phi \;\therefore\; b' = |b^2|^{\frac{1}{2}}\varepsilon^{\frac{i}{3}\phi}\,;\; b'' = |b^2|^{\frac{1}{2}}\varepsilon^{\frac{i}{3}\phi}, \tag{48}$$

then it becomes $0 = 4\cos^3\frac{1}{3}\phi - 3\cos\frac{1}{3}\phi - \cos\phi$, the well known relation between the cosine of an arc and the cosine of its third part. Since an arc determined by (47) has three different thirds, there will be three real roots to the cubic as given by the system

$$0 = ax_1 + b + 2\,|b^2|^{\frac{1}{2}} \cos\frac{\phi}{3}, \tag{49$_1$}$$

$$0 = ax_2 + b + 2\,|b^2|^{\frac{1}{2}} \cos\tfrac{1}{3}(\phi + 2\pi), \tag{49$_2$}$$

$$0 = ax_3 + b + 2\,|b^2|^{\frac{1}{2}} \cos\tfrac{1}{3}(\phi - 2\pi). \tag{49$_3$}$$

For the reciprocal solution assume

$$\cos\psi = |c^3|\,.\,|c^2|^{-\frac{3}{2}}, \tag{50}$$

then

$$0 = 2\,|c^2|^{\frac{1}{2}} \cos\tfrac{1}{3}\psi + c + dx_1^{-1}, \tag{51$_1$}$$

$$0 = 2\,|c^2|^{\frac{1}{2}} \cos\tfrac{1}{3}(\psi - 2\pi) + c + dx_2^{-1}, \tag{51$_2$}$$

$$0 = 2\,|c^2|^{\frac{1}{2}} \cos\tfrac{1}{3}(\psi + 2\pi) + c + dx_3^{-1}. \tag{51$_3$}$$

The connectant is

$$|bc| = |b^2|^{\frac{1}{2}}.\,|c^2|^{\frac{1}{2}} \cos\tfrac{1}{3}(\phi + \psi). \tag{52}$$

To determine the proper ψ to ϕ, we either compute ϕ by (47) and $\frac{1}{3}(\phi + \psi)$ by (52) or else the latter and ψ by (50). We have then in (46)

$$0 = |b^2|^{\frac{1}{2}} \sin\tfrac{1}{3}\phi\,.\,x_1 + |c^2|^{\frac{1}{2}} \sin\tfrac{1}{3}\psi, \tag{53$_1$}$$

$$0 = |b^2|^{\frac{1}{2}} \sin\tfrac{1}{3}(\phi + 2\pi)\,.\,x_2 + |c^2|^{\frac{1}{2}} \sin\tfrac{1}{3}(\psi - 2\pi), \tag{53$_2$}$$

$$0 = |b^2|^{\frac{1}{2}} \sin\tfrac{1}{3}(\phi - 2\pi)\,.\,x_3 + |c^2|^{\frac{1}{2}} \sin\tfrac{1}{3}(\psi + 2\pi). \tag{53$_3$}$$

3. The quartic $0 = ax^4 + 4bx^3 + 6cx^2 + 4dx + e$ admits of being solved in five distinct ways, consistent with the plan of this paper, each corresponding to one of its five coefficients, the a-method being symmetrical to the e-method, the b- to the d- and the c-method being symmetrical to itself.

We have

$$\begin{aligned} 0 &= (ax+b)^4 - 6(b^2 - ac)(ax+b)^2 + 8(b^3 - \tfrac{3}{2}abc + \tfrac{1}{2}a^2d)(ax+b) \\ &\qquad - 3(b^4 - 2ab^2c + \tfrac{4}{3}a^2bd - \tfrac{1}{3}a^3e) \\ &= (ax+b)^4 - 6|b^2|(ax+b)^2 + 8|b^3|(ax+b) - 3|b^4|, \end{aligned} \tag{54}$$

if $\quad b^4 - 2ab^2c + \tfrac{4}{3}a^2bd - \tfrac{1}{3}a^3e = |b^4|$, the quartic variant. (55)

Assuming $$0 = ax + b + b' + b'' + b''', \tag{56}$$

we have, squaring,

$$(ax+b)^2 - (b'^2 + b''^2 + b'''^2) = 2(b'b'' + b''b''' + b'''b').$$

Squaring again, we have

$$\begin{aligned} &(ax+b)^4 - 2(b'^2 + b''^2 + b'''^2)(ax+b)^2 + (b'^2 + b''^2 + b'''^2)^2 \\ &\qquad = 4(b'^2b''^2 + b''^2b'''^2 + b'''^2b'^2) + 8b'b''b'''(b' + b'' + b'''), \end{aligned}$$

hence by (56)

$$\begin{aligned} 0 &= (ax+b)^4 - 2(b'^2 + b''^2 + b'''^2)(ax+b)^2 + 8b'b''b'''(ax+b) \\ &\qquad - \{4(b'^2b''^2 + b''^2b'''^2 + b'''^2b'^2) - (b'^2 + b''^2 + b'''^2)^2\}. \end{aligned} \tag{57}$$

Comparing this with (54), we have

$$|b^2| = \tfrac{1}{3}(b'^2 + b''^2 + b'''^2), \tag{58}$$

$$|b^3| = b'b''b''', \tag{59}$$

$$|b^4| = \tfrac{4}{3}(b'^2b''^2 + b''^2b'''^2 + b'''^2b'^2) - \tfrac{1}{3}(b'^2 + b''^2 + b'''^2)^2, \tag{60}$$

whence $$b'^2 + b''^2 + b'''^2 = 3|b^2|, \tag{61}$$

$$b'^2b''^2 + b''^2b'''^2 + b'''^2b'^2 = \tfrac{9}{4}|b^2|^2 + \tfrac{3}{4}|b^4|, \tag{62}$$

$$b'^2b''^2b'''^2 = |b^3|^2, \tag{63}$$

whence the cubic resolvent

$$\begin{aligned} 0 &= (y - b'^2)(y - b''^2)(y - b'''^2) \\ &= y^3 - 3|b^2|y^2 + \tfrac{3}{4}(3|b^2|^2 + |b^3|)y - |b^3|^2 \\ &= (y - |b^2|)^3 - \tfrac{3}{4}(|b^2|^2 - |b^4|)(y - |b^2|) + \tfrac{1}{4}(|b^2|^3 + 3|b^4|.|b^2| - 4|b^3|^2) \quad (64) \\ &= (y - |b^2|)^3 - \tfrac{3}{4}a^2(c^2 - \tfrac{4}{3}bd + \tfrac{1}{3}ae)(y - |b^2|) \\ &\qquad - \frac{a^3}{4}\{c_3 - (2bd + ae)c + ad^2 + b^2e\}, \end{aligned} \tag{65}$$

or placing

$$\frac{2}{a}(y-|b^2|)=\lambda \tag{66}$$

and denoting

$c^2-\frac{4}{3}bd+\frac{1}{3}ae=(4_2)_0$,[*] the quadric invariant of the quartic, $\quad (67)$

$$-\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix} = c^3-(2bd+ae)c+ad^2+b^2e=(4_3)_0,^{*} \left.\begin{matrix}\text{the cubic invariant}\\ \text{of the quartic,}\end{matrix}\right\} \tag{68}$$

we have the cardinal resolvent of the quartic

$$0=\lambda^3-3(4_2)_0\lambda-2(4_3)_0. \tag{69}$$

The three roots of this are given by the system

$$\lambda_1= \sqrt[3]{(4_3)_0+\sqrt{(4_3)_0^2-(4_2)_0^3}}+ \sqrt[3]{(4_3)_0-\sqrt{(4_3)_0^2-(4_2)_0^3}}, \tag{70$_1$}$$

$$\lambda_2=1^{\frac{1}{3}} \sqrt[3]{(4_3)_0+\sqrt{(4_3)_0^2-(4_2)_0^3}}+1^{-\frac{1}{3}}\sqrt[3]{(4_3)_0-\sqrt{(4_3)_0^2-(4_2)_0^3}}, \tag{70$_2$}$$

$$\lambda_3=1^{-\frac{1}{3}} \sqrt[3]{(4_3)_0+\sqrt{(4_3)_0^2-(4_2)_0^3}}+1^{\frac{1}{3}}\sqrt[3]{(4_3)_0-\sqrt{(4_3)_0^2-(4_2)_0^3}}, \tag{70$_3$}$$

or if all three roots are real,

$$\lambda_1=2(4_2)_0^{\frac{1}{2}}\cos\tfrac{1}{3}\theta, \tag{71$_1$}$$

$$\lambda_2=2(4_2)_0^{\frac{1}{2}}\cos\tfrac{1}{3}(\theta+2\pi), \tag{71$_2$}$$

$$\lambda_3=2(4_2)_0^{\frac{1}{2}}\cos\tfrac{1}{3}(\theta-2\pi), \tag{71$_3$}$$

where

$$\cos\theta=(4_3)_0(4_2)_0^{-\frac{3}{2}}. \tag{72}$$

We have then by (66), using successively the three values of y,

$$b'^2 =|b^2|+\frac{a}{2}\lambda_1=b^2-a\left(c-\frac{\lambda_1}{2}\right), \tag{73$_1$}$$

$$b''^2 =|b^2|+\frac{a}{2}\lambda_2=b^2-a\left(c-\frac{\lambda_2}{2}\right), \tag{73$_2$}$$

$$b'''^2 =|b^2|+\frac{a}{2}\lambda_2=b^2-a\left(c-\frac{\lambda_3}{2}\right), \tag{73$_3$}$$

and we see that this method becomes too much involved in imaginaries in case the resolvent (69) has but one real root, but is very convenient if all three roots are real.

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[*] A notation for covariants in general, which I have proposed in my article, "Symmetries of the Cubic," etc. Thus $(n_p)_g$ denotes a covariant of an n-tic of weight in coefficients $=p$ and of degree $=g$. For an invariant $g=0$. It will be noticed that I have taken $(4_2)_0=\frac{1}{3}S$ and $(4_3)_0=T$ of Salmon. The forms (70) thus become similar to (18) and (29).

The four roots of the quartic are then given by the system

$$0 = ax_1 + b \pm (b' + b'' + b'''), \tag{74_1}$$
$$0 = ax_2 + b \pm (b' - b'' - b'''), \tag{74_2}$$
$$0 = ax_3 + b \pm (-b' + b'' - b'''). \tag{74_3}$$
$$0 = ax_4 + b \pm (-b' - b'' + b''') \quad (\pm \text{ if } |b^3| \text{ is } \pm \text{ by } (59)). \tag{74_4}$$

This method may of course be applied reciprocally by exchanging x for x^{-1}, a for e, b for d, d for b and e for a, and the requisite formulæ may be written down at once.

We have

$$\begin{aligned} 0 &= ex^{-4} + 4dx^{-3} + 6cx^{-2} + 4bx^{-1} + a \\ &= (ex^{-1} + d)^4 - 6\,(d^2 - ce)(ex^{-1} + d)^2 + 8\,(d^3 - \tfrac{3}{2}\,cde + \tfrac{1}{2}\,be^2)(ex^{-1} + d) \\ &\qquad - 3\,(d^4 - 2cde^2 + \tfrac{4}{3}\,bde^2 - \tfrac{1}{3}\,ae^3) \\ &= (ex^{-1} + d)^4 - 6\,|d^2|\,(ex^{-1} + d)^2 + 8\,|d^3|\,(ex^{-1} + d) - 3\,|d^4|, \end{aligned} \tag{75}$$

where

$$d^2 - ce = |d^2|, \text{ the quadric retrovariant of the quartic,} \tag{76}$$
$$d^3 - \tfrac{3}{2}\,cde + \tfrac{1}{2}\,be^2 = |d^3|, \text{ `` cubic `` `` ``} \tag{77}$$
$$d^4 - 2cd^2e + \tfrac{4}{3}\,bde^2 - \tfrac{1}{3}\,ae^3 = |d^4|, \text{ `` quartic `` `` ``} \tag{78}$$

Assuming

$$0 = ex^{-1} + d + d' + d'' + d''', \tag{79}$$

we derive

$$\begin{aligned} 0 &= (ex^{-1} + d)^4 - 2\,(d'^2 + d''^2 + d'''^2)(ex^{-1} + d)^2 + 8d'd''d'''\,(ex^{-1} + d) \\ &\qquad - \{4\,(d'^2d''^2 + d''^2d'''^2 + d'''^2d'^2) - (d'^2 + d''^2 + d'''^2)^2\}, \end{aligned} \tag{80}$$

and comparing this with (75) we derive

$$d'^2 + d''^2 + d'''^2 = 3\,|d^2|, \tag{81}$$
$$d'^2d''^2 + d''^2d'''^2 + d'''^2d'^2 = \tfrac{9}{4}\,|d^2|^2 + \tfrac{3}{4}\,|d^4|, \tag{82}$$
$$d'^2d''^2d'''^2 = |d^3|^2, \tag{83}$$

whence the cubic resolvent

$$0 = (z - d'^2)(z - d''^2)(z - d'''^2), \tag{84}$$
$$= z^3 - 3\,|d^2|\,z^2 + \tfrac{3}{4}\,(3\,|d^2|^2 + |d^4|)\,z - |d^3|^2, \tag{85}$$
$$= (z - |d^2|)^3 - \tfrac{3}{4}\,(|d^2|^2 - |d^4|)(z - |d^2|) + \tfrac{1}{4}\,(|d^2|^3 - 4\,|d^3|^2 + 3\,|d^2|.|d^4|), \tag{86}$$
$$= (z - |d^2|)^3 - \tfrac{3}{4}\,e^2\,(4_2)_0\,(z - |d^2|) - \frac{e^3}{4}\,(4_3)_0, \tag{87}$$

so that assuming

$$z - |d^2| = \frac{e}{2}\lambda, \tag{88}$$

we have again the cardinal resolvent (69).

If this has all its roots real, we may use them in the forms

$$d'^2 = |d^2| + \frac{e}{2}\lambda_1 = d^2 - e\left(c - \frac{\lambda_1}{2}\right), \tag{89_1}$$

$$d''^2 = |d^2| + \frac{e}{2}\lambda_2 = d^2 - e\left(c - \frac{\lambda_2}{2}\right), \tag{89_2}$$

$$d'''^2 = |d^2| + \frac{e}{2}\lambda_3 = d^2 - e\left(c - \frac{\lambda_3}{2}\right), \tag{89_3}$$

and then we have finally the reciprocal root forms

$$0 = ex_1^{-1} + d \pm (d' + d'' + d'''), \tag{90_1}$$

$$0 = ex_2^{-1} + d \pm (d' - d'' - d'''), \tag{90_2}$$

$$0 = ex_3^{-1} + d \pm (-d' + d'' - d'''), \tag{90_3}$$

$$0 = ex_4^{-1} + d \pm (-d' - d'' + d''') \quad (\pm \text{ if } |d^3| \text{ is } \pm). \tag{90_4}$$

These two methods, in which we easily recognize Euler's method generalized, are strictly analogous to the methods we have used for the quadric and cubic; they may be called the methods of completing to the biquadrate. The methods to be given now are methods of completing to a square, and were first published in my article, "A New, Simple and Symmetrical Solution of the Quartic" (Mathematical Magazine, Vol. II, 4). In an improved form they are as follows:

Adding $2(c+\lambda)x^2$ to the second term and deducting it from the third, we have

$$0 = ax^4 + 2(2bx + c + \lambda)x^2 + 4\left(c - \frac{\lambda}{2}\right)x^2 + 4dx + e \tag{91}$$

$$= (ax^2 + 2bx + c + \lambda)^2 - \left\{4\left[b^2 - a\left(c - \frac{\lambda}{2}\right)\right]x^2 \right.$$
$$\left. + 4[b(c+\lambda) - ad]x + (c+\lambda)^2 - ae\right\}. \tag{92}$$

The second term becomes also a square if

$$0 = \left[b^2 - a\left(c - \frac{\lambda}{2}\right)\right]\left[(c+\lambda)^2 - ae\right] - [b(c+\lambda) - ad]^2$$

$$= \begin{vmatrix} a, & b, & c+\lambda \\ b, & c - \frac{\lambda}{2}, & d \\ c+\lambda, & d, & e \end{vmatrix}$$

$$= \lambda^3 - 3(4_2)_0\lambda - 2(4_3)_0,$$

which is the cardinal resolvent (69).

Theoretically, either root of this resolvent, real or imaginary, may be used in (91) to make the second term a square, but since only one of the roots must be used at a time, we shall use only a real root in practice. We have then in (91)

$$0 = ax^2 + 2bx + c + \lambda \pm \left(2x\sqrt{b^2 - a\left(c - \frac{\lambda}{2}\right)} + \sqrt{(c+\lambda)^2 - ae}\right)$$

$$\text{if } b(c+\lambda) > ad, \quad (93_+)$$

or

$$0 = ax^2 + 2bx + c + \lambda \pm \left(2x\sqrt{b^2 - a\left(c - \frac{\lambda}{2}\right)} - \sqrt{(c+\lambda)^2 - ae}\right)$$

$$\text{if } b(c+\lambda) < ad. \quad (93_-)$$

The quadric with the upper sign then gives one pair of roots, and that with the lower sign the remaining two.

Arranging terms differently in (91) we have also

$$0 = 4\left(c - \frac{\lambda}{2}\right)x^2 + 4(bx^2 + d)x + ax^4 + 2(c+\lambda)x^2 + e \quad (94)$$

$$= \left[2\left(c - \frac{\lambda}{2}\right)x + bx^2 + d\right]^2$$

$$-\left\{\left[b^2 - a\left(c - \frac{\lambda}{2}\right)\right]x^4 + 2\left[bd - (c+\lambda)\left(c - \frac{\lambda}{2}\right)\right]x^2 + d^2 - e\left(c - \frac{\lambda}{2}\right)\right\}. \quad (95)$$

The second term becomes a square if

$$0 = \left[b^2 - a\left(c - \frac{\lambda}{2}\right)\right]\left[d^2 - e\left(c - \frac{\lambda}{2}\right)\right] - \left[bd - (c+\lambda)\left(c - \frac{\lambda}{2}\right)\right]^2$$

$$= \begin{vmatrix} a, & b, & c+\lambda \\ b, & c - \frac{\lambda}{2}, & d \\ c+\lambda, & d, & e \end{vmatrix}, \text{ i. e. resolvent (69),}$$

and applying one of its real roots, we have the quadrics

$$0 = bx^2 + 2\left(c - \frac{\lambda}{2}\right)x + d \pm \left(x^2\sqrt{b^2 - a\left(c - \frac{\lambda}{2}\right)} + \sqrt{d^2 - e\left(c - \frac{\lambda}{2}\right)}\right)$$

$$\text{if } bd > (c+\lambda)\left(c - \frac{\lambda}{2}\right), \quad (96_+)$$

$$0 = bx^2 + 2\left(c - \frac{\lambda}{2}\right)x + d \pm \left(x^2\sqrt{b^2 - a\left(c - \frac{\lambda}{2}\right)} - \sqrt{d^2 - e\left(c - \frac{\lambda}{2}\right)}\right)$$

$$\text{if } bd < (c+\lambda)\left(c - \frac{\lambda}{2}\right). \quad (96_-)$$

Arranging terms in (91) differently again we have

$$0 = e + 2(2dx^{-1} + c + \lambda)x^2 + 4\left(c - \frac{\lambda}{2}\right)x^2 + 4bx^3 + ax^4 \tag{97}$$

$$= [e + (2dx^{-1} + c + \lambda)x^2]^2$$
$$-\left\{4\left[d^2 - e\left(c - \frac{\lambda}{2}\right)\right]x^2 + 4[d(c+\lambda) - be]x^3 + [(c+\lambda)^2 - ae]x^4\right\}. \tag{98}$$

Here the second term becomes a square if

$$0 = \left[d^2 - e\left(c - \frac{\lambda}{2}\right)\right][(c+\lambda)^2 - ae] - [d(c+\lambda) - be]^2$$
$$= \begin{vmatrix} a\ , & b\ , & c+\lambda \\ b\ , & c - \frac{\lambda}{2}, & d \\ c+\lambda, & d\ , & e \end{vmatrix}, \text{ i. e. resolvent (69)},$$

and applying one of its real roots, we have

$$0 = ex^{-2} + 2dx^{-1} + c + \lambda \pm \left(2x^{-1}\sqrt{d^2 - e\left(c - \frac{\lambda}{2}\right)} + \sqrt{(c+\lambda)^2 - ae}\right)$$
$$\text{if } d(c+\lambda) > be, \tag{99$_+$}$$
$$0 = ex^{-2} + 2dx^{-1} + c + \lambda \pm \left(2x^{-1}\sqrt{d^2 - e\left(c - \frac{\lambda}{2}\right)} - \sqrt{(c+\lambda)^2 - ae}\right)$$
$$\text{if } d(c+\lambda) < be. \tag{99$_-$}$$

These forms result also if in (93) we write x^{-1} for x, e for a and d for b. Doing this also in (96), it will remain the same.

We easily deduce from (74)

$$b = -\frac{a}{4}(x_1 + x_2 + x_3 + x_4) \tag{100}$$

$$\pm b' = -\frac{a}{4}(x_1 + x_2 - x_3 - x_4) \tag{100$'$}$$

$$\pm b'' = -\frac{a}{4}(x_1 - x_2 + x_3 - x_4) \tag{100$''$}$$

$$\pm b''' = -\frac{a}{4}(x_1 - x_2 - x_3 + x_4) \quad (\pm \text{ if } |b^3| \text{ is} \pm). \tag{100$'''$}$$

Since

$$c - \frac{\lambda_1}{2} = \frac{b^2 - b'^2}{a}; \quad c - \frac{\lambda_2}{2} = \frac{b^2 - b''^2}{a}; \quad c - \frac{\lambda_3}{2} = \frac{b^2 - b'''^2}{a}, \tag{101}$$

we have

$$c - \frac{\lambda_1}{2} = \frac{a}{4}(x_1 + x_2)(x_3 + x_4) = \frac{e}{4}(x_1^{-1} + x_2^{-1})(x_3^{-1} + x_4^{-1}), \qquad (102_1)$$

$$c - \frac{\lambda_2}{2} = \frac{a}{4}(x_1 + x_3)(x_3 + x_4) = \frac{e}{4}(x_1^{-1} + x_3^{-1})(x_2^{-1} + x_4^{-1}), \qquad (102_2)$$

$$c - \frac{\lambda_3}{2} = \frac{a}{4}(x_1 + x_4)(x_2 + x_3) = \frac{e}{4}(x_1^{-1} + x_4^{-1})(x_2^{-1} + x_3^{-1}), \qquad (102_3)$$

therefore, since $6c = a(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)$, we have also

$$c + \lambda_1 = \frac{a}{2}(x_1x_2 + x_3x_4) = \frac{e}{2}(x_1^{-1}x_2^{-1} + x_3^{-1}x_4^{-1}), \qquad (103_1)$$

$$c + \lambda_2 = \frac{a}{2}(x_1x_3 + x_2x_4) = \frac{e}{2}(x_1^{-1}x_3^{-1} + x_2^{-1}x_4^{-1}), \qquad (103_2)$$

$$c + \lambda_3 = \frac{a}{2}(x_1x_4 + x_2x_3) = \frac{e}{2}(x_1^{-1}x_4^{-1} + x_2^{-1}x_3^{-1}), \qquad (103_3)$$

and placing

$$\sqrt{(c + \lambda_1)^2 - ae} = 2c'; \quad \sqrt{(c + \lambda_2)^2 - ae} = 2c''; \quad \sqrt{(c + \lambda_3)^2 - ae} = 2c''', \qquad (104)$$

we have also

$$c' = \frac{a}{4}(x_1x_2 - x_3x_4) = -\frac{e}{4}(x_1^{-1}x_2^{-1} - x_3^{-1}x_4^{-1}), \qquad (105_1)$$

$$c'' = \frac{a}{4}(x_1x_3 - x_2x_4) = -\frac{e}{4}(x_1^{-1}x_3^{-1} - x_2^{-1}x_4^{-1}), \qquad (105_2)$$

$$c''' = \frac{a}{4}(x_1x_4 - x_2x_3) = -\frac{e}{4}(x_1^{-1}x_4^{-1} - x_2^{-1}x_3^{-1}). \qquad (105_3)$$

From (90) we derive similarly

$$d = -\frac{e}{4}(x_1^{-1} + x_2^{-1} + x_3^{-1} + x_4^{-1})$$
$$= -\frac{a}{4}(x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2 + x_1x_2x_3), \qquad (106)$$

$$\pm d' = -\frac{e}{4}(x_1^{-1} + x_2^{-1} - x_3^{-1} - x_4^{-1})$$
$$= -\frac{a}{4}(x_2x_3x_4 + x_3x_4x_1 - x_4x_1x_2 - x_1x_2x_3), \qquad (106')$$

$$\pm d'' = -\frac{e}{4}(x_1^{-1} + x_2^{-1} + x_3^{-1} - x_4^{-1})$$
$$= -\frac{a}{4}(x_2x_3x_4 - x_3x_4x_1 + x_4x_1x_2 - x_1x_2x_3), \qquad (106'')$$

$$\pm d''' = -\frac{e}{4}(x_1^{-1} - x_2^{-1} - x_3^{-1} + x_4^{-1})$$
$$= -\frac{a}{4}(x_2x_3x_4 - x_3x_4x_1 - x_4x_1x_2 + x_1x_2x_3)\ (\pm \text{ if } |d^3| \text{ is } \pm). \qquad (106''')$$

We have then

$$b \pm b' = -\frac{a}{2}(x_1 + x_2);\quad b \mp b' = -\frac{a}{2}(x_3 + x_4), \tag{107$_1$}$$

$$b \pm b'' = -\frac{a}{2}(x_1 + x_3);\quad b \mp b'' = -\frac{a}{2}(x_2 + x_4), \tag{107$_2$}$$

$$b \pm b''' = -\frac{a}{2}(x_1 + x_4);\quad b \mp b''' = -\frac{a}{2}(x_2 + x_3)(\pm \text{ if } |b^3| \text{ is } \pm), \tag{107$_3$}$$

and

$$c + \lambda_1 \pm 2c' = ax_1x_2;\quad c + \lambda_1 \mp 2c' = ax_3x_4(\pm \text{ if } b(c + \lambda_1) - ad \text{ is } \pm), \tag{108$_1$}$$
$$c + \lambda_2 \pm 2c'' = ax_1x_3;\quad c + \lambda_2 \mp 2c'' = ax_2x_4(\pm \text{ if } b(c + \lambda_2) - ad \text{ is } \pm), \tag{108$_2$}$$
$$c + \lambda_3 \pm 2c''' = ax_1x_4;\quad c + \lambda_3 \mp 2c''' = ax_2x_3(\pm \text{ if } b(c + \lambda_3) - ad \text{ is } \pm), \tag{108$_3$}$$

whence the root forms

$$0 = ax_1 + b \pm b' \pm\sqrt{(b \pm b')^2 - a(c + \lambda_1 \pm 2c')}, \tag{109$_1$}$$
$$0 = ax_2 + b \pm b' \mp\sqrt{(b \pm b')^2 - a(c + \lambda_1 \pm 2c')}, \tag{109$_2$}$$
$$0 = ax_3 + b \mp b' \pm\sqrt{(b \mp b')^2 - a(c + \lambda_1 \mp 2c')}. \tag{109$_3$}$$
$$0 = ax_4 + b \mp b' \mp\sqrt{(b \mp b')^2 - a(c + \lambda_1 \mp 2c')}, \tag{109$_4$}$$

where the sign of b' is the same as that of $|b^3|$, and the sign of c' as that of $b(c+\lambda_1) - ad$. By pairing the roots differently, we can write out other root-forms. These can however conveniently only be used if all the roots of the quartic are real. If either one pair or both pair of roots are conjugate imaginaries, there will be only one manner of pairing so that the roots can be obtained from (109) by real operations. We have thus a criterion between the case of four real and four imaginary roots, that for the first the auxiliaries b', b'', b''', or c', c'', c''', or d', d'', d''' are all real, while in the other case only one set, b', c', d', say are real.

From (96) we derive the root-forms

$$0 = (b \mp b')x_1 + c - \frac{\lambda_1}{2} \pm\sqrt{\left(c - \frac{\lambda_1}{2}\right)^2 - (b \mp b')(d \pm d')}, \tag{110$_1$}$$

$$0 = (b \mp b')x_2 + c - \frac{\lambda_1}{2} \mp\sqrt{\left(c - \frac{\lambda_1}{2}\right)^2 - (b \mp b')(d \pm d')}, \tag{100$_2$}$$

$$0 = (b \pm b')x_3 + c - \frac{\lambda_1}{2} \pm\sqrt{\left(c - \frac{\lambda_1}{2}\right)^2 - (b \pm b')(d \mp d')}, \tag{110$_3$}$$

$$0 = (b \pm b')x_4 + c - \frac{\lambda_1}{2} \mp\sqrt{\left(c - \frac{\lambda_1}{2}\right)^2 - (b \pm b')(d \mp d')}, \tag{110$_4$}$$

where b' has the sign of $|b^3|$ and d' that of $bd - (c + \lambda_1)\left(c - \frac{\lambda_1}{2}\right)$.

Changing in (109) and (110) x^{-1} for x, e for a and d for b, we have other equivalent root-forms.

From (108) we easily derive the following interesting root-forms:

$$ae^{\frac12}x_1 = (c + \lambda_1 \pm 2c')^{\frac12}(c + \lambda_2 \pm 2c'')^{\frac12}(c + \lambda_3 \pm 2c''')^{\frac12} = a^{\frac12}ex_1^{-1}, \tag{111$_1$}$$
$$ae^{\frac12}x_2 = (c + \lambda_1 \pm 2c')^{\frac12}(c + \lambda_2 \mp 2c'')^{\frac12}(c + \lambda_3 \mp 2c''')^{\frac12} = a^{\frac12}ex_2^{-1}, \tag{111$_2$}$$
$$ae^{\frac12}x_3 = (c + \lambda_1 \mp 2c')^{\frac12}(c + \lambda_2 \pm 2c'')^{\frac12}(c + \lambda_3 \mp 2c''')^{\frac12} = a^{\frac12}ex_3^{-1}, \tag{111$_3$}$$
$$ae^{\frac12}x_4 = (c + \lambda_1 \mp 2c')^{\frac12}(c + \lambda_2 \mp 2c'')^{\frac12}(c + \lambda_3 \pm 2c''')^{\frac12} = a^{\frac12}ex_4^{-1}, \tag{111$_4$}$$

which can be used conveniently only if all roots are real or imaginary.

Assuming

$$\cos\beta = \frac{b}{\sqrt{a\left(c - \frac{\lambda}{2}\right)}}; \quad \cos\gamma = \frac{c + \lambda}{\sqrt{ae}}; \quad \cos\delta = \frac{d}{\sqrt{e\left(c - \frac{\lambda}{2}\right)}},^* \tag{112}$$

then the cardinal resolvent (69) becomes

$$0 = \begin{vmatrix} a, & \sqrt{a\left(c - \frac{\lambda}{2}\right)}\cos\beta, & \sqrt{ae}\cos\gamma \\ \sqrt{a\left(c - \frac{\lambda}{2}\right)}\cos\beta, & c - \frac{\lambda}{2}, & \sqrt{e\left(c - \frac{\lambda}{2}\right)}\cos\delta \\ \sqrt{ae}\cos\gamma, & \sqrt{e\left(c - \frac{\lambda}{2}\right)}\cos\delta, & e \end{vmatrix}$$

$$= \begin{vmatrix} 1, & \cos\beta, & \cos\gamma \\ \cos\beta, & 1, & \cos\delta \\ \cos\gamma, & \cos\delta, & 1 \end{vmatrix}$$

$$= 1 - \cos^2\beta - \cos^2\gamma - \cos^2\delta + 2\cos\beta\cos\gamma\cos\delta, \tag{113}$$

which is satisfied by the system

$$0 = \beta - \gamma - \delta,$$

or

$$0 = \delta - \beta - \gamma,$$

or

$$0 = \gamma - \delta - \beta. \tag{114}$$

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*These angles were first assumed in my article, "A New, Simple and Symmetrical Solution of the Quartic" (Math. Magazine, Vol. II, 4).

We have, then, the following elegant expressions for the quadric factors of the quartic:

$$0=\left[a^{\frac{1}{2}}x^2+2\left(c-\frac{\lambda}{2}\right)^{\frac{1}{2}}x\varepsilon^{\beta i}+e^{\frac{1}{2}}\varepsilon^{\pm\gamma i}\right]$$

$$\times\left[a^{\frac{1}{2}}x^2+2\left(c-\frac{\lambda}{2}\right)^{\frac{1}{2}}x\varepsilon^{-\beta i}+e^{\frac{1}{2}}\varepsilon^{\mp\gamma i}\right](\pm \text{ if } \cos\beta\,\cos\gamma \gtrless \cos\delta), \quad (93')$$

$$0=\left[\alpha^{\frac{1}{2}}x^2\varepsilon^{-\beta i}+2\left(c-\frac{\lambda}{2}\right)^{\frac{1}{2}}x+e^{\frac{1}{2}}\varepsilon^{\mp\delta i}\right]$$

$$\times\left[a^{\frac{1}{2}}x^2\varepsilon^{\beta i}+2\left(c-\frac{\lambda}{2}\right)^{\frac{1}{2}}x+e^{\frac{1}{2}}\varepsilon^{\pm\delta i}\right](\pm \text{ if } \cos\delta\,\cos\beta \gtrless \cos\gamma), \quad (96')$$

$$0=\left[a^{\frac{1}{2}}x^2\varepsilon^{\gamma i}+2\left(c-\frac{\lambda}{2}\right)^{\frac{1}{2}}x\varepsilon^{\pm\delta i}+e^{\frac{1}{2}}\right]$$

$$\times\left[a^{\frac{1}{2}}x^2\varepsilon^{-\gamma i}+2\left(c-\frac{\lambda}{2}\right)^{\frac{1}{2}}x\varepsilon^{\mp\delta i}+e^{\frac{1}{2}}\right](\pm \text{ if } \cos\gamma\,\cos\delta \gtrless \cos\beta). \quad (99')$$

The arcs β, γ, δ can be real only if the auxiliaries b', c', d' or b'', c'', d'' or b''', c''', d''' are pure imaginaries. This cannot be if the roots of the quartic are real or if two roots are imaginary, and in case of four imaginary roots, supposing x_1 conjugate to x_2 and x_3 to x_4, then b', c', d' are real and b'', c'', d'' as well as b''', c''', d''' are pure imaginaries. System (74) becomes then

$$0=ax_1+b\pm b'\pm bi(\tan\beta_2+\tan\beta_3), \quad (74_1')$$
$$0=ax_2+b\pm b'\mp bi(\tan\beta_2+\tan\beta_3), \quad (74_2')$$
$$0=ax_3+b\mp b'\pm bi(\tan\beta_2-\tan\beta_3), \quad (74_3')$$
$$0=ax_4+b\mp b'\mp bi(\tan\beta_2-\tan\beta_3)\ (\pm \text{ if } |b^3| \text{ is } \pm), \quad (74_4')$$

where we may write

$$\tan\beta_2\pm\tan\beta_3=\frac{\sin(\beta_2\pm\beta_3)}{\cos\beta_2\,\cos\beta_3},$$

which formulæ are specially adapted to the third case of the quartic. In the same manner (90) for the reciprocal solution become

$$0=ex_1^{-1}+d\pm d'\pm di(\tan\delta_2+\tan\delta_3), \quad (90_1')$$
$$0=ex_2^{-1}+d\pm d'\mp di(\tan\delta_2+\tan\delta_3), \quad (90_2')$$
$$0=ex_3^{-1}+d\mp d'\pm di(\tan\delta_2-\tan\delta_3), \quad (90_3')$$
$$0=ex_4^{-1}+d\mp d'\mp di(\tan\delta_2-\tan\delta_3)\ (\pm \text{ if } |d^3| \text{ is } \pm). \quad (90_4')$$

Also (111) take the form

$$x_1 = \left(\frac{c + \lambda_1 + 2c'}{a}\right)^{\frac{1}{4}} \varepsilon^{\frac{i}{2}(\gamma_2 + \gamma_3)}, \tag{$111'_1$}$$

$$x_2 = \left(\frac{c + \lambda_1 + 2c'}{a}\right)^{\frac{1}{4}} \varepsilon^{-\frac{i}{2}(\gamma_2 + \gamma_3)}, \tag{$111'_2$}$$

$$x_3 = \left(\frac{c + \lambda_1 - 2c'}{a}\right)^{\frac{1}{4}} \varepsilon^{\frac{i}{2}(\gamma_2 - \gamma_3)}, \tag{$111'_3$}$$

$$x_4 = \left(\frac{c + \lambda_1 - 2c'}{a}\right)^{\frac{1}{4}} \varepsilon^{-\frac{i}{2}(\gamma_2 - \gamma_3)}, \tag{$111'_4$}$$

for which we may write the more symmetrical forms

$$x_1 = \left(\frac{e}{a}\right)^{\frac{1}{4}} \varepsilon^{i(\gamma_1 + \gamma_2 + \gamma_3)}, \tag{$111''_1$}$$

$$x_2 = \left(\frac{e}{a}\right)^{\frac{1}{4}} \varepsilon^{\frac{i}{2}(\gamma_1 - \gamma_2 - \gamma_3)}, \tag{$111''_2$}$$

$$x_3 = \left(\frac{e}{a}\right)^{\frac{1}{4}} \varepsilon^{\frac{i}{2}(-\gamma_1 + \gamma_2 - \gamma_3)}, \tag{$111''_3$}$$

$$x_4 = \left(\frac{e}{a}\right)^{\frac{1}{4}} \varepsilon^{\frac{i}{2}(-\gamma_1 - \gamma_2 + \gamma_3)}, \tag{$111''_4$}$$

4. The covariants of the cubic and quartic have interesting relations to the auxiliaries which we have employed. For the cubic we have

$$\begin{aligned}(3_2)_2 = \begin{vmatrix} ax + b, & bx + c \\ bx + c, & cx + d \end{vmatrix} = \begin{vmatrix} 1, & x, & x^2 \\ a, & b, & c \\ b, & c, & d \end{vmatrix} &= -(b^2 - ac)\,x^2 - (ad - bc)\,x - (c^2 - bd) \\ &= -|b^2|\,x - 2\,|bc|\,x - |c^2| \\ &= -b'b''x + (b'c' + b''c'')\,x - c'c'' \\ &= -(b'x - c'')(b''x - c'),\end{aligned} \tag{112}$$

therefore $-(3_2)_2$ may be considered a combination of the quadric variant $|b^2|$ with the quadric retrovariant. Also

$$\begin{aligned}(3_3)_3 &= 2\,(b^3 - \tfrac{3}{2}abc + \tfrac{1}{2}a^2d)\,x^2 + 3\,(b^2c - 2ac^2 + abd)\,x^2 \\ &\qquad - 3\,(bc^2 - 2b^2d + acd)\,x - 2\,(c^3 - \tfrac{3}{2}bcd + \tfrac{1}{2}ad^2) \\ &= (b'^3 + b''^3)\,x^3 - 3\,(b'^2c'' + b''^2c')\,x^2 + 3\,(b'c''^2 + b''c'^2)\,x - (c''^3 + c'^3) \\ &= (b'x - c'')^3 + (b''x - c')^3,\end{aligned} \tag{113}$$

therefore $\frac{1}{2}(3_3)_3$ is a combination of the cubic variant $|b^3|$ with the cubic retrovariant $|c^3|$ and Cayley's relation

$$(3_3)_3^2 + 4\,(3_2)_2^3 = \Delta_3\,(3_1)_3^2 \tag{114}$$

appears as a combination of the relations

$$4\{\,|b^3|^2 - |b^2|^3\} = a^2\Delta_3,$$
$$4\{\,|c^3|^2 - |c^2|^3\} = d^2\Delta_3,$$

and as I have shown in my paper, "Symmetries of the Cubic," etc., to reduce the cubic

$$0 = ax^3 + 3bx^2y + 3cxy^2 + dy^3 \tag{115}$$

to its canonical form

$$0 = \frac{1}{\Delta_3^{\frac{1}{2}}}\left[(b'x - c''y)^3 - (b''x - c'y)^3\right], \tag{45'}$$

we must assume the linear transformation

$$x = c''X + c'Y \quad \therefore \quad X = -\frac{b''x - c'}{b'c' - b''c''}, \tag{116}$$

$$y = b'X + b''Y \therefore \quad Y = \frac{b'x - c''}{b'c' - b''c''}, \tag{117}$$

or for one variable,

$$x = \frac{c''X + c'}{b'X + b''} \quad \therefore \quad X = -\frac{b''x - c'}{b'x - c''}. \tag{118}$$

For the quartic we have the quartic covariant

$$(4_2)_4 = \begin{vmatrix} ax^2 + 2bx + c, & bx^2 + 2cx + d \\ bx^2 + 2cx + d, & cx^2 + 2dx + e \end{vmatrix}$$
$$= (ac - b^2)\,x^4 + 2\,(ad - bc)\,x^3 + (ae + 2bd - 3c^2)\,x^2$$
$$+ 2\,(be - cd)\,x + ce - d^2. \tag{119}$$

Deducting this from

$$\frac{\lambda_1}{2}(4_1)_4 = a\frac{\lambda_1}{2}x^4 + 4b\frac{\lambda_1}{2}x^3 + 6c\frac{\lambda_1}{2}x^2 + 4d\frac{\lambda_1}{2}x + e\frac{\lambda_1}{2}$$

we obtain

$$\frac{\lambda_1}{2}(4_1)_4 - (4_2)_4 = \left[b^2 - a\left(c - \frac{\lambda_1}{2}\right)\right]x^4 + 2\,[b\,(c + \lambda_1) - ad]\,x^3$$
$$+ [3\,(c + \lambda_1)\,c - 2bd - ae]\,x^2 + 2\,[d\,(c + \lambda_1) - be]\,x + \left[d^2 - e\left(c - \frac{\lambda_1}{2}\right)\right]$$
$$= b'^2x^4 \pm 4b'c'x^3 + 2\,(2c'^2 + b'd')\,x^2 \pm 4c'd'x + d'^2$$
$$= (b'x^2 \pm 2c'x + d'^2)^2, \tag{120'}$$

and similarly

$$\frac{\lambda_2}{2}(4_1)_4 - (4_2)_4 = (b''x^2 \pm 2c''x + d'')^2, \tag{120''}$$

$$\frac{\lambda_3}{2}(4_1)_4 - (4_2)_4 = (b'''x^2 \pm 2c'''x + d''')^2. \tag{120'''}$$

Adding, we have

$$-3(4_2)_4 = (b'x^2 \pm 2c'x + d')^2 + (b''x^2 \pm 2c''x + d'')^2 + (b'''x^2 \pm 2c'''x + d''')^2. \quad (121)$$

This form being analogous to (61) and (81), is a sort of combination of the two. We have also

$$\begin{aligned}
&\left[\frac{\lambda_1}{2}(4_1)_4 - (4_2)_4\right]\left[\frac{\lambda_2}{2}(4_1)_4 - (4_2)_4\right] \\
&\quad + \left[\frac{\lambda_2}{2}(4_1)_4 - (4_2)_4\right]\left[\frac{\lambda_3}{2}(4_1)_4 - (4_2)_4\right] + \left[\frac{\lambda_2}{2}(4_1)_4 - (4_2)_4\right]\left[\frac{\lambda_1}{2}(4_1)_4 - (4_2)_4\right] \\
&= \tfrac{1}{4}(4_1)_4^2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) - 2(4_1)_4(4_2)_4(\lambda_1 + \lambda_2 + \lambda_3) + 3(4_2)_4^2 \\
&= -\tfrac{3}{4}(4_1)_4^2(4_2)_0 + 3(4_2)_4^2 = (b'x^2 \pm 2c'x + d')^2(b''x^2 \pm 2c''x + d'')^2 \\
&\quad + (b''x^2 \pm 2c''x + d'')^2(b'''x^2 \pm 2c'''x + d''')^2 \\
&\quad + (b'''x^2 \pm 2c'''x + d''')^2(b'x^2 \pm 2c'x + d')^2; \qquad (122)
\end{aligned}$$

also

$$\begin{aligned}
&\left[\frac{\lambda_1}{2}(4_1)_4 - (4_2)_4\right]\left[\frac{\lambda_2}{2}(4_1)_4 - (4_2)_4\right]\frac{\lambda_3}{2}(4_1)_4 - (4_2)_4\Big] \\
&= \tfrac{1}{8}(4_1)_4^3\lambda_1\lambda_2\lambda_3 - \tfrac{1}{4}(4_2)_4(4_1)_4^2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) + \tfrac{1}{2}(4_2)_4^2(4_1)_4(\lambda_1 + \lambda_2 + \lambda_3) - (4_2)_4^3 \\
&= \tfrac{1}{4}(4_1)_4^3(4_3)_0 + \tfrac{3}{4}(4_2)_4(4_1)_4^2(4_2)_0 - (4_2)_4^3 = \tfrac{1}{4}(4_3)_6^2 \text{ by Cayley's relation.} \qquad (123)
\end{aligned}$$

Let us write for brevity

$$\begin{aligned}
b'x^2 \pm 2c'x + d' &= |b', c', d'|, \\
b''x^2 \pm 2c''x + d'' &= |b'', c'', d''|, \\
b'''x^2 \pm 2c'''x + d''' &= |b''', c''', d'''|,
\end{aligned}$$

then if $|b, c, d|$ is the general representative of each, we have the cubic in $|b, c, d|^2$:

$$\begin{aligned}
0 &= (|b, c, d|^2 - |b', c', d'|^2)(|b, c, d|^2 - |b'', c'', d''|^2)(|b, c, d|^2 - |b''', c''', d'''|^2) \\
&= |b, c, d|^6 + 3(4_2)_4|b, c, d|^4 + 3[(4_2)_4^2 - \tfrac{1}{4}(4_2)_0(4_1)_4^2]\,.\,|b, c, d|^2 - \tfrac{1}{4}(4_3)_6^2 \\
&= (|b, c, d|^2 + (4_2)_4)^3 - \tfrac{3}{4}(4_2)\,.\,(4_1)_4^2(|b, c, d|^2 \\
&\qquad + (4_2)_4) - \tfrac{1}{4}(4_3)_6^2 - (4_2)_4^3 + \tfrac{3}{4}(4_2)_0(4_1)_4^2(4_2)_4, \qquad (124)
\end{aligned}$$

and if in this we put

$$\frac{\lambda}{2} = \frac{|b, c, d|^2 + (4_2)}{(4_1)_4}, \qquad (125)$$

and $\quad \tfrac{1}{4}(4_3)_6^2 = -(4_2)_4^3 + \tfrac{3}{4}(4_2)_0(4_1)_4^2(4_2)_4 + \tfrac{1}{4}(4_3)_0(4_1)_4^3$ (Cayley's relation),

it will become the cardinal resolvent (69).

Combining (120) in pairs, we have the following expressions for the quartic:

$$(4_1)_4 = \frac{2}{\lambda_1 - \lambda_2}\left(|b',\, c',\, d'|^2 - |b'',\, c'',\, d''|^2\right), \tag{126_1}$$

$$= \frac{2}{\lambda_2 - \lambda_3}\left(|b'',\, c'',\, d''|^2 - |b''',\, c''',\, d'''|^2\right), \tag{126_2}$$

$$= \frac{2}{\lambda_3 - \lambda_1}\left(|b''',\, c''',\, d'''|^2 - |b',\, c',\, d'|^2\right).^* \tag{126_3}$$

Taking the mean of these we have since

$$(\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2 = \tfrac{1}{16}\Delta_4 \text{ (the discriminant)}, \tag{127}$$

$$(4_1)_4 = \frac{8}{3\Delta_4^{\frac{1}{2}}}\left\{(\lambda_2 - \lambda_3)|b',\, c',\, d'|^2 + (\lambda_3 - \lambda_1)|b'',\, c'',\, d''|^2 + (\lambda_1 - \lambda_2|b''',\, c''',\, d'''|^2\right\}. \tag{128}$$

Each of (126) can be resolved in two quadratic factors, which yield each a pair of roots to the quartic.

5. I conclude by giving the general solutions of the four quantics in their homogeneous form.

$$0 = (1_1)_1 = ax + by = (1_1)\wedge(x,\, y), \tag{129}$$

$$0 = \frac{ax + by}{\eta x + \xi y}, \tag{130}$$

$$0 = (2_1)_2 = ax^3 + 2bxy + cy^2 = (2_1)\wedge(x,\, y)_2^2, \tag{131}$$

$$\left.\begin{aligned} 0 &= \frac{(ax_1 + by_1)\,\xi + (bx_1 + cy_1)\,\eta}{\eta x_1 - \xi y} + b', \\ 0 &= \frac{(ax_2 + by_2)\,\xi + (bx_2 + cy_2)\,\eta}{\eta x_2 - \xi y_2} - b', \end{aligned}\right\} \tag{132}$$

$$b' = \sqrt{b^2 - ac} = \sqrt{|b^2|}, \tag{7}$$

$$0 = (3_1)_3 = ax^3 + 3bx^2y + 3cxy^2 + dy^3 = (3_1)\wedge(x,\, y)_3, \tag{133}$$

$$\left.\begin{aligned} 0 &= \frac{(ax_1 + by_1)\,\xi^2 + 2\,(bx_1 + cy_1)\,\xi\eta + (cx_1 + dy_1)\,\eta^2}{\eta x_1 - \xi y_1} \\ &\qquad - (b'\xi - c''\eta) - (b''\xi - c''\eta) \\ 0 &= \frac{(ax_2 + by_2)\,\xi^2 + 2\,(bx_2 + cy_2)\,\xi\eta + (cx_2 + dy_2)\,\eta^2}{\eta x_2 - \xi y_2} \\ &\qquad - (b'\xi - c''\eta)\,1^{\frac{1}{3}} - (b''\xi - c''\eta)\,1^{-\frac{1}{3}}, \\ 0 &= \frac{(ax_3 + by_3)\,\xi^2 + 2\,(bx_3 + cy_3)\,\xi\eta + (cx_3 + dy_3)\,\eta^2}{\eta x_3 - \xi y_3} \\ &\qquad - (b'\xi - c''\eta)\,1^{-\frac{1}{3}} - (b''\xi - c'\eta)\,1^{\frac{1}{3}}, \end{aligned}\right\} \tag{134}$$

$$b' = \sqrt[3]{|b^3| + \sqrt{|b^3|^2 - |b^2|^3}};\quad b'' = \sqrt[3]{|b^3| - \sqrt{|b^3|^2 - |b^2|^3}}, \tag{18}$$

$$c' = \sqrt[3]{|c^3| \pm \sqrt{|c^3|^2 - |c^2|^3}};\quad c'' = \sqrt[3]{|c^3| \mp \sqrt{|c^3|^2 - |c^2|^3}}. \tag{29}$$

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*Credited to Darboux by Matthiesen in his great work, "Grundzüge der antiken und modernen Algebra," page 750.

Sign criterion,

$$b'c' + b''c'' = 2|bc| = ad - bc, \tag{35}$$

$$0 = (4_1)_4 = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 = (4_1)^\wedge(x,\ y)_4, \tag{135}$$

$$\left.\begin{aligned}
0 &= \frac{(ax_1 + by_1)\,\xi^3 + 3\,(bx_1 + cy_1)\,\xi^2\eta + 3\,(cx_1 + dy_1)\,\xi\eta^2 + (dx_1 + ey_1)\,\eta^3}{\eta x_1 - \xi y_1} \\
&\quad \pm \left[b'\xi^2 \pm 2c'\xi\eta + d'\eta^2 + b''\xi^2 \pm 2c''\xi\eta + d''\eta^2 + b'''\xi^2 \pm 2c'''\xi\eta + d'''\eta^2\right], \\
0 &= \frac{(ax_2 + by_2)\,\xi^3 + 3\,(bx_2 + cy_2)\,\xi^2\eta + 3\,(cx_2 + dy_2)\,\xi\eta^2 + (dx_2 + ey_2)\,\eta^3}{\eta x_2 - \xi y} \\
&\quad \pm \left[b'\xi^2 \pm 2c'\xi\eta + d'\eta^2 - (b''\xi^2 \pm 2c''\xi\eta + d''\eta^2) - (b'''\xi^2 \pm 2c'''\xi\eta + d''\eta^2\right], \\
0 &= \frac{(ax_3 + by_3)\,\xi^3 + 3\,(bx_3 + cy_3)\,\xi^2\eta + 3\,(cx_3 + dy_3)\,\xi\eta^2 + (dx_3 + ey_3)\,\eta_3}{\eta x_3 - \xi y_3} \\
&\quad \pm \left[-(b'\xi^2 \pm 2c'\xi\eta + d'\eta^2) + b''\xi^2 \pm 2c''\xi\eta + d''\eta^2 - (b'''\xi^2 \pm 2c'''\xi\eta + d''\eta^2)\right], \\
0 &= \frac{(ax_4 + by_4)\,\xi^3 + 3\,(bx_4 + cy_4)\,\xi^2\eta + 3\,(cx_4 + dy_4)\,\xi\eta^2 + (dx_4 + ey_4)\,\eta^3}{\eta x_4 - \xi y_4} \\
&\quad \pm \left[-(b'\xi^2 \pm 2c'\xi\eta + d'\eta^2) - (b''\xi^2 \pm 2c''\xi\eta + d''\eta^2) + b'''\xi^2 \pm 2c'''\xi\eta + d'''\eta^2\right].
\end{aligned}\right\} \tag{136}$$

Cardinal resolvent,

$$0 = \lambda^3 - 3\,(4_2)_0\,\lambda - 2\,(4_3)_0, \tag{69}$$

$$b' = \sqrt{b^2 - a\left(c - \frac{\lambda_1}{2}\right)};\quad b'' = \sqrt{b^2 - a\left(c - \frac{\lambda_2}{2}\right)};$$

$$b''' = \sqrt{b^2 - a\left(c - \frac{\lambda_3}{2}\right)};\quad b'b''b''' = |b^3|, \tag{73}$$

$$2c' = \sqrt{(c + \lambda_1)^2 - ae};\quad 2c'' = \sqrt{(c + \lambda_2)^2 - ae};$$

$$2c''' = \sqrt{(c + \lambda_3)^2 - ae};\ \text{to be taken} \pm \text{if } b\,(c + \lambda) - ad \text{ is } \pm, \tag{104}$$

$$d' = \sqrt{d^2 - e\left(c - \frac{\lambda_1}{2}\right)};\quad d'' = \sqrt{d^2 - e\left(c - \frac{\lambda_2}{2}\right)};$$

$$d''' = \sqrt{d^2 - e\left(c - \frac{\lambda_3}{2}\right)};\quad d'd''d''' = |d^3|. \tag{89}$$

Each of these solutions reverts to the respective quantic itself if $\eta x = \xi y$. If we put $\eta = 0$ we have the direct solutions, and if $\xi = 0$ we have the reciprocal. Since the ratio $\frac{\xi}{\eta}$ is perfectly arbitrary, these forms admit of infinite variation.

The notion of expressing the roots in this manner is due to Clebsch, who restricted the first term to the type

$$\frac{(ax+by)\,\xi^n + \binom{n}{1}(bx+cy)\,\xi^{n-1} + \ldots\ldots + lx + my}{\eta x - \xi y}, \tag{90}$$

which reverts to the quantic $= 0$ if $\frac{x}{y} = \frac{\xi}{\eta}$.

The forms for the general roots given above for the cubic differ from those given by Matthiesen, page 730, who has in the second term in place of $-(b'\xi - c''\eta) - (b''\xi - c'\eta)$ the expression $+\sqrt[3]{(3_3)_3^{\wedge}(\xi, \eta)}$, supposing that ξ/η is a root to $(3_2)_2^{\wedge}(\xi, \eta)$. Thus the form is not truly general as mine is.

If we do not restrict ourselves to type (90), we can easily form a general root-form of as many variables as there are distinct solutions of a quantic by simply multiplying each by an indeterminate factor and adding them.